BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 8

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CONTENTS

I. REVIEW

Last time we:

(1) Defined the genus of a compact, connected Riemann surface X as $\dim_{\mathbb{C}} \Omega(X)$, the dimension of the space of holomorphic differentials; defined the canonical map
(2) Defined triangulations and proved the Riemann-Hurwitz Theorem
(3) Defined the fundamental group

II. THE FUNDAMENTAL GROUP AND COVERING SPACES

II.1. A crash course on the fundamental group.

Definition 1. Let X be a topological space and $P \in X$. Two loops γ_0 and γ_1 based at P are (pointed) homotopic if there is a continuous map $H : [0,1] \times [0,1] \to X$ such that

$$H|_{0\times[0,1]} = \gamma_0, \qquad H|_{1\times[0,1]} = \gamma_1$$

and $H(s,0) = H(s,1) = P$ for all $s \in [0,1]$. Such an H is a (pointed) homotopy of γ_0 and γ_1.

A homotopy defines a continuous family of loops $\gamma_s(t) := H(s,t)$ interpolating γ_0 and γ_1.

Lemma 2. *Homotopy is an equivalence relation on the set of loops based at P.*

Given a path γ, we denote its homotopy class by $[\gamma]$.

Definition 3. Given two paths γ_1, γ_2 with $\gamma_1(1) = P = \gamma_2(0)$, we define their concatenation by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2 \\ \gamma_2(2t-1) & \text{if } 1/2 \le t \le 1\,. \end{cases}$$

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In other words, $\gamma_1 * \gamma_2$ is the path γ_1 followed by the path γ_2, with the variable appropriately rescaled so that the domain is still $[0,1]$.

Definition 4. Let X be a topological space and $P \in X$. The fundamental group of X (based at P) is the set of homotopy classes of loops based at P, and is denoted $\pi_1(X,P)$. The space X is simply connected if $\pi_1(X,P)$ if it is connected and $\pi_1(X,P)$ is the trivial group for some (hence, any) choice of basepoint $P \in X$.

Proposition 5. *The fundamental group is a group under the concatenation operation defined above. The identity element is $[c_P]$, where c_P is the constant map sending $t \mapsto P$ for all $t \in [0,1]$. The inverse of an element $[\gamma]$ is the class of the reverse map $\gamma^{-1}(t) := \gamma(1-t)$.*

Lemma 6. *Assume X is path connected and let $P, Q \in X$. Then $\pi_1(X,P) \cong \pi_1(X,Q)$.*

Proof idea. Let $\alpha : [0,1] \to X$ be a path from P to Q, so $\alpha(0) = P$ and $\alpha(1) = Q$. Given a loop γ based at P, then $\alpha^{-1} * \gamma * \alpha$ is a loop based at Q. Similarly, given a loop δ based at Q, we obtain a loop $\alpha * \delta * \alpha^{-1}$ based at P. These two maps are homomorphisms and mutually inverse up to homotopy, hence provide the desired isomorphism. □

Example 7.

(1) Let $X = \mathbb{S}^1$. We'll show this rigorously later using covering spaces, but intuitively we would already guess that $\pi_1(\mathbb{S}^1, 1) \xrightarrow{\sim} \mathbb{Z}$, via the map that takes a loop γ to its winding number, i.e., the number of times it goes around the origin.

(2) Let $\Lambda \subseteq \mathbb{C}$ be a lattice and $X = \mathbb{C}/\Lambda$ be the corresponding torus. Note that X is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. Again, just thinking intuitively we see that $\pi_1(X,x) \cong \mathbb{Z} \times \mathbb{Z}$. The generators $(1,0)$ and $(0,1)$ correspond to the path that traverses one of the longitudinal circles, and the path that traverses one of the meridional circles, respectively.

Remark 8. In fact, it is true in general that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. The idea is that a loop $\gamma : [0,1] \to X \times Y$ is equivalent to the data of loops $\delta_1 : [0,1] \to X$ and $\delta_2 : [0,1] \to Y$, and this correspondence respects homotopy.

Remark 9. There's much more to say about the fundamental group. The fundamental group defines a functor from the category of topological spaces to the category of groups, so a continuous map $f : (X,x) \to (Y,y)$ of pointed topological spaces induces a map $f_* : \pi_1(X,x) \to \pi_1(Y,y)$, given by $[\gamma] \mapsto [f \circ \gamma]$, where $\gamma : [0,1] \to X$ is a loop.

II.2. **Covering spaces.** Covering spaces provide a powerful tool for computing fundamental groups. They are also in some sense (which can be made very precise) the topological analogue of an algebraic closure, so studying covering spaces is like a topological version of Galois theory.

Historically, they often arose when people were trying to solve differential equations on a space that had "holes", i.e., was not simply connected. Often the problem couldn't be solved on the starting space, but did have a solution after passing to a suitable cover.

Definition 10. Let X be a topological space. A covering space of X is a topological space E together with a continuous map $\pi : E \to X$ called a covering map such that the following property holds. For each $P \in X$ there exists a neighborhood V of P such that

$\pi^{-1}(V) = \bigsqcup_i U_i$, where the sets U_i are pairwise disjoint and the restriction $\pi|_{U_i} \to V$ is a homeomorphism. We say that such a neighborhood V is evenly covered by π.

Example 11. Let $X = \mathbb{S}^1 \subseteq \mathbb{C}$ be the circle, considered as the set of points z with $|z| = 1$. Then

$$\begin{aligned} \pi : X &\to X \\ z &\mapsto z^2 \end{aligned}$$

is a covering space. [Give proof by picture using plot.]

Example 12. Consider again the circle $X = \mathbb{S}^1 \subseteq \mathbb{C}$. Then

$$\begin{aligned} \pi : \mathbb{R} &\to \mathbb{S}^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

is a covering space of X. (One can visualize $\mathbb{R}$ embedded in $\mathbb{R}^3$ as a helix with p the projection map down to the plane.) [See picture on p. 49 of GGD.]

Remark 13. The fibers $\pi^{-1}(P)$ of a covering are discrete, since $\pi^{-1}(U) = \bigsqcup_I U_i$ is a disjoint union, where U is an evenly covered neighborhood of P.

Remark 14. If X is a Riemann surface, then E inherits a unique holomorphic structure such that the covering map $\pi : E \to X$ is holomorphic. The idea is that we simply pull back the charts of X to E: given a chart (U, φ) on X, define a chart $(\pi^{-1}(U), \varphi \circ \pi)$ on E.

Definition 15. Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be coverings of X. A morphism from π_1 to π_2 is a continuous map $f : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ f$, i.e., such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & X & \end{array}$$

The map f is an isomorphism of coverings if it is a homeomorphism.

Definition 16. A deck transformation of a covering $\pi : E \to X$ is an automorphism of the covering. The set of deck transformations of π is a group, denoted $\mathrm{Deck}(E/X)$ or $\mathrm{Deck}\left(E \xrightarrow{\pi} X\right)$.

Theorem 17. *Let X be a connected Riemann surface. Then there exists a covering $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ connected and simply connected. Moreover $\widetilde{X}$ is unique up to isomorphism.*

Definition 18. The covering space $\widetilde{X}$ in the previous theorem is called the universal covering space of X.

Theorem 19. *Let X be a connected Riemann surface and $p : \widetilde{X} \to X$ be its universal cover. Then* $\mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X, x)$ *for any choice of basepoint $x \in X$.*

Example 20.

- The covering map $p : \mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ is the universal cover since $\mathbb{R}$ is simply connected. Given $t \in \mathbb{S}^1$, we can write $t = e^{2\pi i \theta}$ for some $\theta \in \mathbb{R}$. Then
$$p^{-1}(e^{2\pi i \theta}) = \{\theta + n : n \in \mathbb{Z}\} .$$
Thus the deck transformations $\alpha \in \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1)$ are all of the form $\alpha : s \mapsto s + n$ for some $n \in \mathbb{Z}$, so
$$\pi_1(\mathbb{S}^1, 1) \cong \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1) \cong \mathbb{Z} .$$
- Let $\Lambda \subseteq \mathbb{C}$ be a lattice and $X = \mathbb{C}/\Lambda$ be the corresponding torus. Then the quotient map $p : \mathbb{C} \to X$ is the universal cover, since $\mathbb{C}$ is simply connected. [Show pictures at
`https://angyansheng.github.io/blog/my-final-year-project-i` and
`https://www.math3ma.com/blog/a-recipe-for-the-universal-cover-of-x-y`.]
Choose generators ω_1, ω_2 for Λ, so $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Given a point $x \in X$, then the fiber above x is
$$p^{-1}(x) = x + \Lambda = \{x + m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} .$$
Then the deck transformations $\alpha \in \mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda)$ are all of the form $\alpha : z \mapsto z + \lambda$ for some $\lambda \in \Lambda$, i.e.,
$$\alpha : z \mapsto z + m\omega_1 + n\omega_2$$
for some $m, n \in \mathbb{Z}$. Thus
$$\pi_1(X, x) \cong \mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda) \cong \Lambda \cong \mathbb{Z} \oplus \mathbb{Z} .$$

Covering spaces possess some important lifting properties, one of which is the path-lifting lemma.

Lemma 21 (Path-lifting lemma)**.** *Let $p : E \to X$ be a covering space. Let γ be a path on X and let $x = \gamma(0)$. Given any preimage $e \in p^{-1}(x)$ there exists a unique path $\widetilde{\gamma}$ on E such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = e$.*

Definition 22. Such a $\widetilde{\gamma}$ is called a lift of γ based at e.

Since $\widetilde{\gamma}$ projects to the loop γ under p, then $\widetilde{\gamma}(1)$ must be in the fiber $p^{-1}(x)$, too. Thus the path-lifting lemma allows us to define an action of $\pi_1(X, x)$ on a fiber $p^{-1}(x)$ of a covering $p : \widetilde{X} \to X$.

Given $[\gamma] \in \pi_1(X, x)$ and $\widetilde{x} \in p^{-1}(x)$, let $\widetilde{\gamma}$ be the unique lift of γ with $\widetilde{\gamma}(0) = \widetilde{x}$. Define
$$\widetilde{x} \cdot [\gamma] := \widetilde{\gamma}(1) .$$

Lemma 23. *The above definition*
$$\begin{aligned} p^{-1}(x) \times \pi_1(X, x) &\to p^{-1}(x) \\ (\widetilde{x}, [\gamma]) &\mapsto \widetilde{x} \cdot [\gamma] = \widetilde{\gamma}(1) \end{aligned}$$
gives a right $\pi_1(X, x)$ action on $p^{-1}(x)$.

Proof. Note that, due to our convention that $\gamma_1 * \gamma_2$ traverses first γ_1, then γ_2, then

$$\widetilde{x} \cdot [\gamma_1 * \gamma_2] = (\widetilde{x} \cdot [\gamma_1]) \cdot [\gamma_2]$$

so this is a *right* action. □

As you probably recall from group theory, actions of a group G on a set X are in bijective correspondence with group homomorphisms $G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the symmetric group on X, i.e., the set of all bijections $X \to X$. However, in order for this to give a homomorphism and not an anti-homomorphism, we actually require a *left* action. (That is, unless you also have your permutations act on the right, too...)

Fortunately, we can convert this right action into a left action by defining

$$[\gamma] \circ \widetilde{x} := \widetilde{x} \cdot [\gamma^{-1}].$$

With this definition, the "associativity" condition of a group action holds:

$$[\gamma_1 * \gamma_2] \circ \widetilde{x} = \widetilde{x} \cdot [(\gamma_1 * \gamma_2)^{-1}] = \widetilde{x} \cdot [\gamma_2^{-1} * \gamma_1^{-1}] = (\widetilde{x} \cdot [\gamma_2^{-1}]) \cdot [\gamma_1^{-1}] = [\gamma_1] \circ ([\gamma_2] \circ \widetilde{x}).$$

Thus we get a group homomorphism $\pi_1(X, x) \to \mathrm{Sym}(p^{-1}(x))$. If the fiber $p^{-1}(x)$ is finite, containing d points, then by labeling the points $1, 2, \ldots, d$, we can identify $\mathrm{Sym}(p^{-1}(x)) \cong S_d$, hence we obtain a homomorphism $\pi_1(X, x) \to S_d$.

Definition 24. Let X be a connected Riemann surface, $x \in X$ and let $p : E \to X$ be a covering space. Let $\theta : \pi_1(X, x) \to \mathrm{Sym}(p^{-1}(x))$ be the group homomorphism defined above. Then θ is called the monodromy representation of p and the image of θ is called its monodromy group.

II.3. **Groups acting on Riemann surfaces.**

Definition 25. Let G be a group.

- Let X be a topological space. A (continuous) action of G on X is a group homomorphism $G \to \mathrm{Homeo}(X)$, the group of self-homeomorphisms $X \to X$.
- Let X be a Riemann surface and G be a group. A (holomorphic) action of G on X is a group homomorphism $G \to \mathrm{Aut}(X)$.

Given a group G acting on a Riemann surface X, we can form the quotient space $G \backslash X$ whose points are the G-orbits of X. There is a natural quotient map

$$\pi : X \to G \backslash X$$
$$x \mapsto [x]$$

where $[x]$ denotes the G-orbit of x. Without further restrictions, $G \backslash X$ will only be a topological space, not necessarily a Riemann surface. The following properties of group actions yield nice properties of the quotient space $G \backslash X$ and the quotient map π.

Definition 26. Let G be a group acting (holomorphically) on a Riemann surface X.

(a) The action is faithful (or effective) if the kernel of the homomorphism $G \to \mathrm{Aut}(X)$ is trivial.

(b) The action is free if for all points $x \in X$, the stabilizer

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\}$$

is trivial.

(c) The action is properly discontinuous or wandering if, for each $x \in X$ there exists an open neighborhood $U \ni x$ such that the set

$$\{g \in G : gU \cap U \neq \varnothing\}$$

is finite. In particular, this means that $\mathrm{Stab}_G(x)$ is finite for all $x \in X$.

Lemma 27. *If G acts on X properly discontinuously, then $G \backslash X$ is Hausdorff.*

Proposition 28. *Suppose G is a group acting on X freely and properly discontinuously. Then the quotient map $\pi : X \to G\backslash X$ is a covering map with deck transformation group G.*

Theorem 29. *Let G be a finite group acting faithfully on a Riemann surface X. Then $G\backslash X$ can be given the structure of a Riemann surface. Moreover, the quotient map $\pi : X \to G\backslash X$ is holomorphic of degree $\#G$ and $e_P(\pi) = \#\mathrm{Stab}_G(P)$ for all $P \in X$.*

We now have the language to define the Galois correspondence given by the universal covering space.

Theorem 30. *Let X be a connected Riemann surface and $p : \widetilde{X} \to X$ be its universal cover.*

(a) The action of $\mathrm{Deck}(\widetilde{X}/X)$ on $\widetilde{X}$ is free and properly discontinuous. Moreover, the action is transitive on each fiber.

(b) The action induces an isomorphism of Riemann surfaces

$$\begin{aligned} \mathrm{Deck}(\widetilde{X}/X)\backslash\widetilde{X} &\to X \\ [\widetilde{x}] &\mapsto p(\widetilde{x})\,. \end{aligned}$$

(c) Let $q : E \to X$ be a covering. Then there exists a subgroup $H \leq \mathrm{Deck}(\widetilde{X}/X)$ such that $E \cong H\backslash\widetilde{X}$ as Riemann surfaces, and the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\sim} & H\backslash\widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow \\ X & \xrightarrow{\sim} & \mathrm{Deck}(\widetilde{X}/X)\backslash\widetilde{X} \end{array}$$

Remark 31. Parts (b) and (c) of the above theorem should remind you of Galois theory.

$$\begin{array}{ccc} K & \qquad\qquad & 1 \\ | & & | \\ E = K^H & & H \\ | & & | \\ F & & \mathrm{Gal}(K/F) \end{array}$$

$$
\begin{array}{ccc}
\widetilde{X} & \qquad\qquad & 1 \\
\downarrow & & | \\
E = H\backslash\widetilde{X} & & H \\
\downarrow & & | \\
X & & \mathrm{Deck}(\widetilde{X}/X)
\end{array}
$$

Example 32. Consider the universal cover $p : \mathbb{R} \to \mathbb{S}^1$, $p : t \mapsto e^{2\pi i t}$ and the covering $q : \mathbb{S}^1 \to \mathbb{S}^1$, $q : z \mapsto z^2$. Let $H \leq \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1)$ be the subgroup of all deck transformations of the form $\alpha : s \mapsto s + 2m$ for some $m \in \mathbb{Z}$. Then

$$2\mathbb{Z} \cong H \leq \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1) \cong \mathbb{Z}$$

so $[\mathrm{Deck}(\mathbb{R}/\mathbb{S}^1) : H] = 2$, and $H\backslash\mathbb{R} \cong 2\mathbb{Z}\backslash\mathbb{R} \cong \mathbb{S}^1$. (We're identifying every other loop in the helix, so we get 2 coils of the helix after quotienting.) Moreover, we get a commutative diagram.

$$
\begin{array}{cccc}
z & \mathbb{S}^1 & \xrightarrow{\sim} & H\backslash\mathbb{R} = 2\mathbb{Z}\backslash\mathbb{R} \\
\big\downarrow & {\scriptstyle q}\big\downarrow & & \big\downarrow \\
z^2 & \mathbb{S}^1 & \xrightarrow{\sim} & \mathbb{Z}\backslash\mathbb{R}
\end{array}
$$